Period 3, Oct 21, 2024

~~6~~, ~~8~~, ~~12~~, ~~24~~, ~~46~~, ~~36~~, ~~18~~, 20, ~~16~~, ~~40~~

6) $x^2y + y^2x = -2$

$h(x) = g(x)\cdot F(x)$

$h'(x) = g'(x)\cdot F(x) + g(x)\cdot F'(x)$

$2x\cdot y + x^2\cdot\frac{dy}{dx} + 2y\frac{dy}{dx}\cdot x + y^2\cdot 1 = 0$

$\cancel{2xy} + x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} + \cancel{y^2} = 0$

$-2xy \qquad -y^2 \quad -2xy - y^2$

$$\frac{\frac{dy}{dx}\cancel{(x^2+2xy)}}{\cancel{(x^2+2xy)}} = \frac{-2xy-y^2}{(x^2+2xy)} = \frac{dy}{dx} = \frac{-1(2xy+y^2)}{(x^2+2xy)}$$

8 $\sqrt{xy} = x^2y + 1$

$\sqrt{xy} = \sqrt{x}\cdot\sqrt{y} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

$x^{\frac{1}{2}}y^{\frac{1}{2}} = x^2y + 1$

$\frac{1}{2}x^{\frac{1}{2}-1}\cdot y^{\frac{1}{2}} + x^{\frac{1}{2}}\cdot\frac{1}{2}y^{\frac{1}{2}-1}\cdot\frac{dy}{dx} = 2xy + x^2\cdot\frac{dy}{dx} + 0$

$\cancel{\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}} + \frac{x^{\frac{1}{2}}}{2y^{\frac{1}{2}}}\frac{dy}{dx} = 2xy + \cancel{x^2\frac{dy}{dx}}$

$\cancel{-\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}} \quad -x^2\frac{dy}{dx} \quad -\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}} \quad \cancel{-x^2\frac{dy}{dx}}$

$$\frac{\frac{dy}{dx}\left(\frac{\sqrt{x}}{2\sqrt{y}} - x^2\right)}{\left(\frac{\sqrt{x}}{2\sqrt{y}} - x^2\right)} = \frac{\frac{2xy\cdot 2\sqrt{x}}{1\cdot 2\sqrt{x}} - \frac{\sqrt{y}}{2\sqrt{x}}}{\left(\frac{\sqrt{x}}{2\sqrt{y}} - \frac{x^2\cdot 2\sqrt{y}}{1\cdot 2\sqrt{y}}\right)} = \frac{\frac{4xy\sqrt{x} - \sqrt{y}}{2\sqrt{x}}}{\frac{\sqrt{x} - 2x^2\sqrt{y}}{2\sqrt{y}}}$$

$$\frac{(4xy\sqrt{x}-\sqrt{y})}{\cancel{2}\sqrt{x}} \cdot \frac{\cancel{2}\sqrt{y}}{(\sqrt{x}-2x^2\sqrt{y})} = \frac{4xy\sqrt{xy}-y}{x-2x^2\sqrt{xy}}$$

12

$$(\sin \pi x + \cos \pi y)^2 = 2$$

$y = \sin \pi x \Rightarrow y = \sin u$

$u = \pi x$

$\frac{du}{dx} = \pi \qquad \frac{dy}{du} = \cos u$

$\frac{d\cancel{u}}{dx} \cdot \frac{dy}{d\cancel{u}} = \frac{dy}{dx}$

$\pi \cdot \cos u = \pi \cdot \cos \pi x$

$u = \sin \pi x + \cos \pi y$

$$\frac{du}{dx} = \pi \cos \pi x + -\pi \sin \pi y \frac{dy}{dx}$$

$u^2 = 2$

$2u\frac{du}{dx} = 0$

$$\frac{\cancel{2(\sin \pi x + \cos \pi y)}\left(\pi \cos \pi x - \pi \sin \pi y \frac{dy}{dx}\right)}{\cancel{2(\sin \pi x + \cos \pi y)}} = \frac{0}{2(\sin \pi x + \cos \pi y)}$$

$$\cancel{\pi \cos \pi x} - \pi \sin \pi y \frac{dy}{dx} = 0$$
$$\cancel{-\pi \cos \pi x} \qquad -\pi \cos \pi x$$

$$\frac{-\pi \sin \pi y \frac{dy}{dx}}{-\pi \sin \pi y} = \frac{-\cancel{\pi} \cos \pi x}{-\cancel{\pi} \sin \pi y} = \frac{\cos \pi x}{\sin \pi y}$$

16.

$x = \sec\frac{1}{y} = \sec y^{-1}$

$\sec\theta = \frac{1}{\cos\theta}$

$1 = \sec\frac{1}{y}\cdot\tan\frac{1}{y}\cdot\frac{d}{dx}[y^{-1}]$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$1 = \sec\frac{1}{y}\tan\frac{1}{y}\cdot -1\cdot y^{-1-1}\frac{dy}{dx}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\cos\frac{1}{y}\cdot 1 = \cancel{\sec\frac{1}{y}}\,\cancel{\tan\frac{1}{y}}\cdot\left(\cancel{\frac{-1}{y^2}}\right)\frac{dy}{dx}\cdot\cancel{\cos\frac{1}{y}}$ $\cot\frac{1}{y}$ $\cdot\cancel{-y^2}$

$\cot\frac{1}{y}$

$\cdot -y^2$

$$\left(\cos\frac{1}{y}\cot\frac{1}{y}\right)(-y^2) = \frac{dy}{dx}$$

18, 20

$x^2 + y^2 - 4x + 6y + 9 = 0$

$-9 \quad -9$

$1x^2 - 4x + 4 + 1y^2 + 6y + 9 = -9 + 9 + 4$

$a = 1$

$b = -4$

$\boxed{\frac{b}{2} = \frac{-4}{2} = -2}$

$\left(\frac{b}{2}\right)^2 = (-2)^2 = 4$

$a = 1$

$b = 6$

$\frac{b}{2} = \frac{6}{2} = 3$

$\left(\frac{b}{2}\right)^2 = (3)^2 = 9$

$(x-2)^2 + (y+3)^2 = 4$

circle

center$(2,-3)$ $r = \sqrt{4} = 2$

$$(x-2)^2+(y+3)^2=4$$
$$-(x-2)^2 \qquad -(x-2)^2$$
$$\sqrt{(y+3)^2}=\sqrt{4-(x-2)^2}$$
$$y+3=\pm\sqrt{4-(x-2)^2}$$
$$y=\pm\sqrt{4-(x-2)^2}-3$$
$$\boxed{y=\pm(4-(x-2)^2)^{\frac{1}{2}}-3}$$
$$\frac{dy}{dx}=\pm\frac{1}{2\sqrt{4-(x-2)^2}}(0-2(x-2)')=\frac{-\cancel{2}(x-2)}{\pm\cancel{2}\sqrt{4-(x-2)^2}}$$

$$x^2+y^2-4x+6y+9=0$$
$$2x+2y\frac{dy}{dx}-4+6\frac{dy}{dx}=0$$
$$-2x \quad +4 \qquad -2x+4$$
$$\frac{dy}{dx}(2y+6)=\frac{-2x+4}{2y+6}=\frac{-\cancel{2}(x-2)}{\cancel{2}(y+3)}=\frac{-(x-2)}{\sqrt{4-(x-2)^2}\ \cancel{-3+3}}$$
$$2y+6$$

Same

$$16y^2-x^2=16 \Rightarrow 32y\frac{dy}{dx}-2x=0 \Rightarrow \cancel{32y}\frac{dy}{dx}=\frac{2x}{32y} \Rightarrow \frac{dy}{dx}=\frac{x}{16y}$$
$$\frac{\cancel{16}y^2}{\cancel{16}}=\frac{16+x^2}{16}$$
$$\frac{dy}{dx}=\frac{\pm x}{16\cdot\frac{1}{4}(16+x^2)^{\frac{1}{2}}}=\frac{x}{4(16+x^2)^{\frac{1}{2}}}$$
$$\sqrt{y^2}=\sqrt{\frac{16+x^2}{16}}=\frac{\sqrt{16+x^2}}{\sqrt{16}}=\frac{\pm\sqrt{16+x^2}}{4}=y=\frac{\pm 1}{4}(16+x^2)^{\frac{1}{2}}$$

$y = \pm\frac{1}{4}(16+x^2)^{\frac{1}{2}} \Rightarrow y = \pm\frac{1}{4}u^{\frac{1}{2}}$

$u = 16+x^2$

$\frac{du}{dx} = 2x$

$\frac{dy}{du} = \pm\frac{1}{4}\cdot\frac{1}{2}u^{\frac{1}{2}-1} = \frac{\pm 1}{8\sqrt{u}}$

Same

$\frac{du}{dx}\cdot\frac{dy}{du} = 2x\left(\pm\frac{1}{8\sqrt{16+x^2}}\right) = \frac{\pm x}{4\sqrt{16+x^2}}$

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24. $(x+y)^3 = x^3+y^3 \quad (-1,1)$

$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 3x^2+3y^2\frac{dy}{dx}$

$3(x+y)^2\cdot 1 + 3(x+y)^2\frac{dy}{dx} = 3x^2+3y^2\frac{dy}{dx}$

$3(-1+1)^2+3(-1+1)^2\frac{dy}{dx} = 3(-1)^2+3(1)^2\frac{dy}{dx}$

$0+3\cdot 0\cdot\frac{dy}{dx} = 3+3\frac{dy}{dx}$

$0 = 3+3\frac{dy}{dx}$

$-1 = \frac{dy}{dx}$

$(x+y)^3 = x^3+y^3$

$\cancel{x^3}+3x^2y+3xy^2+\cancel{y^3} = \cancel{x^3}+\cancel{y^3}$

$3x^2y+3xy^2 = 0$

$6x\cdot y+3x^2\frac{dy}{dx}+3y^2+3x\cdot 2y\frac{dy}{dx}$

$(-1,1)$

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36)

$7x^2 - 6\sqrt{3}\, xy + 13y^2 - 16 = 0$ $\qquad (\sqrt{3}, 1)$

$14x - 6\sqrt{3}\cdot y + -6\sqrt{3}x \cdot \frac{dy}{dx} + 26y\frac{dy}{dx} + 0 = 0$

$14\sqrt{3} - 6\sqrt{3}\cdot 1 - 6\sqrt{3}\cdot\sqrt{3}\frac{dy}{dx} + 26\frac{dy}{dx} = 0$

$8\sqrt{3} - 6\cdot 3\frac{dy}{dx} + 26\frac{dy}{dx} = 0$

-18

$8\sqrt{3} + 8\frac{dy}{dx} = 0 \quad -8\sqrt{3}$ $\Rightarrow \cancel{8}\frac{dy}{dx} = -8\sqrt{3}$

$-8\sqrt{3}$ $\qquad \cancel{8} \qquad \cancel{4}$

$\frac{dy}{dx} = -\sqrt{3}$

$y - 1 = -\sqrt{3}(x - \sqrt{3})$

$y - \cancel{1} = -\sqrt{3}x + 3$

$+1 \qquad +1$

$y = -\sqrt{3}x + 4$

40.

$y^2(x^2 + y^2) = 2x^2$ $\qquad (1, 1)$

$2y\frac{dy}{dx}(x^2 + y^2) + y^2\left(2x + 2y\frac{dy}{dx}\right) = 4x$

$2\cdot 1\frac{dy}{dx}(1^2 + 1^2) + 1^2\left(2(1) + 2(1)\frac{dy}{dx}\right) = 4(1)$

$4\frac{dy}{dx} + \cancel{2} + 2\frac{dy}{dx} = 4$

$-\cancel{2} \qquad -2$

$4\frac{dy}{dx} + 2\frac{dy}{dx} = 2 \Rightarrow \cancel{6}\frac{dy}{dx} = 2 \Rightarrow \frac{dy}{dx} = \frac{1}{3}$

$6 \qquad 6$

$y - 1 = \frac{1}{3}(x - 1)$

$$x^2y^2 - 2x = 3$$

$$2xy^2 + x^2\cdot 2y\frac{dy}{dx} - 2 = 0$$

$$-2xy^2 \qquad +2 \quad -2xy^2+2$$

$$\frac{2x^2y\frac{dy}{dx}}{2x^2y} = \frac{-2xy^2+2}{2x^2y}$$

$$\frac{dy}{dx} = \frac{\cancel{2}(-xy^2+1)}{\cancel{2}(x^2y)}$$

$$\frac{d^2y}{dx^2} = \frac{[(-1)y^2 + -x\cdot 2y\frac{dy}{dx}) + 0][x^2y] - (-xy^2+1)(2x\cdot y + x^2\frac{dy}{dx})}{(x^2y)^2}$$

$$\frac{d^2y}{dx^2} = \left(-y^2 - 2xy\left(\frac{-x^2y+1}{x^2y}\right)\right)(x^2y) - (1-xy^2)\left(2xy + \frac{x^2(-xy^2+1)}{x^2+y}\right)$$

clean up

Keep going

$$\sin^{-1}x = \arcsin x = y$$

$$\sin y = x$$

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1. $y = e^{\cos x}$ $\Rightarrow y = e^{u}$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$\frac{dy}{du} = e^{u}$

$\frac{du}{dx} \cdot \frac{dy}{du} = \frac{dy}{dx}$

$(-\sin x)(e^{u}) = -\sin x \cdot e^{\cos x} = -e^{\cos x} \sin x$

4. $y = e^{\tan(5-7x^2)}$

$u = \tan(5-7x^2)$

$\frac{dy}{dx} = -14x \sec^2(5-7x^2)\, e^{\tan(5-7x^2)}$

$L = 5-7x^2$

$y = \sin^{-1} x \Rightarrow x = \sin y$

$\sin y = \sin(\sin^{-1} x)$

$\sin y = x$

$y = \cos^{-1} x$

$\cos y = x$

$-\sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx} = \frac{1}{-\sin y}$

$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$

$\sin^2 y + \cos^2 y = 1$

$\sin^2 y = 1 - \cos^2 y$

$\sqrt{\sin^2 y} = \sqrt{1-x^2}$

$\sin y = \pm\sqrt{1-x^2}$

Derivatives of Inverse Trig Functions

$\frac{d}{dx}[\sin^{-1}x] = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}[\cos^{-1}x] = \frac{-1}{\sqrt{1-x^2}}$
$\frac{d}{dx}[\tan^{-1}x] = \frac{1}{1+x^2}$	$\frac{d}{dx}[\cot^{-1}x] = \frac{-1}{1+x^2}$
$\frac{d}{dx}[\sec^{-1}x] = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}[\csc^{-1}x] = \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

- $\sin^{-1}x$ and $\arcsin x$ are the same thing. Both refer to the inverse sine function.

$\cot^2 y + 1 = \csc^2 y$

4) $y = \cot^{-1} e^{2x}$ $\Rightarrow \cot y = e^{2x}$

$\cot y = \cot(\cot^{-1} e^{2x})$

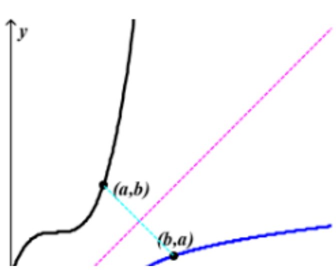
$\cot y = e^{2x}$

$-\csc^2 y \frac{dy}{dx} = 2e^{2x} \Rightarrow \frac{dy}{dx} = \frac{2e^{2x}}{-e^{4x}-1} = \frac{2e^{2x}}{-1(e^{4x}+1)} = \frac{-2e^{2x}}{1+e^{4x}}$

Derivative of an Inverse Function at (x, y)

$$(f^{-1})'(x) = \frac{1}{f'(y)}$$

The derivative of $f^{-1}(x)$ at the point (p,q) is the reciprocal derivative of $f(x)$ at (q,p)



$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Student Example 1

Let $f(x) = x^3 - 1$ find $\left(f^{-1}\right)'(26)$.

$F^{-1}(26) =$

$26 = x^3 - 1$ Find x

$F(0) = 0^3 - 1 = -1$

$F(1) = 1^3 - 1 = 0$

$F(2) = 2^3 - 1 = 8 - 1 = 7$

$F(3) = 3^3 - 1 = 27 - 1 = 26$

$F(-1) = (-1)^3 - 1 = -1 - 1 = -2$

$F(-2) = (-2)^3 - 1 = -9$

$F(-3)$

$F(3) = 26$

$F^{-1}(26) = 3$

$$(f^{-1})'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

$$(F^{-1})'(26) = \frac{1}{F'\left(F^{-1}(26)\right)}$$

$$(F^{-1})'(26) = \frac{1}{F'(3)} = \frac{1}{27}$$

$F(x) = x^3 - 1$

$F'(x) = 3x^2$

$F'(3) = 3(3)^2 = 3 \cdot 9 = 27$

Let $f(x) = \frac{1}{4}x^3 + x - 1$. Find $(f^{-1})'(x)$ when $x = 3$.

$F'(x) = \frac{3}{4}x^2 + 1 \quad F'(2) = \frac{3}{4} \cdot 4 + 1 = 3 + 1 = 4$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \Rightarrow (F^{-1})'(3) = \frac{1}{F'(F^{-1}(3))} = \frac{1}{F'(2)}$$

$$= \frac{1}{4}$$

$F^{-1}(3) = a$

$F(a) = 3 \Rightarrow 3 = \frac{1}{4}a^3 + a - 1$ Find a

Try $0, 1, -1, 2, -2, 3, -3$

$a = 2$

$\frac{1}{4}(2)^3 + 2 - 1 = \frac{1}{4} \cdot 8 + 2 - 1 = 2 + 2 - 1 = 3$

$F^{-1}(3) = 2$

Let f be the function defined by $f(x) = x^3 + x$

If $g(x) = f^{-1}(x)$ and $g(2) = 1$, what is the value of $g'(2)$?

$(F^{-1})'(2)$

$F(a) = 2 \qquad x^3 + x = 2$

$x = 1$

$F^{-1}(2) = a = 1$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$F'(x) = 3x^2 + 1$

$F'(1) = 3 \cdot 1 + 1 = 4$

$$= \frac{1}{F'(F^{-1}(2))}$$

$$(F^{-1})'(2) = \frac{1}{F'(1)} = \frac{1}{4}$$

a) $y = \sin^{-1}(x^2)$

$\sin y = x^2$

$\cos y \frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{\cos y} = \frac{2x}{\sqrt{1-x^4}}$

$\sin^2 y + \cos^2 y = 1$

$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-(x^2)^2} = \sqrt{1-x^4}$